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Supersymmetric Analogs of the Gordon-Andrews Identities, and Related TBA Systems

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Abstract

The Gordon-Andrews identities, which generalize the Rogers-Ramanujan-Schur identities, provide product and fermionic forms for the characters of the minimal conformal field theories (CFTs) $\mathcal{M}(2,2k+1)$. We discuss/conjecture identities of a similar type, providing two different fermionic forms for the characters of the models $\mathcal{SM}(2,4k)$ in the minimal series of N=1 super-CFTs. These two forms are related to two families of thermodynamic Bethe Ansatz (TBA) systems, which are argued to be associated with the $\hat{\phi}_{1,3}^{\rm top}$- and $\hat{\phi}_{1,5}^{\rm bot}$-perturbations of the models $\mathcal{SM}(2,4k)$. Certain other q-series identities and TBA systems are also discussed, as well as a possible representation-theoretical consequence of our results, based on Andrews's generalization of the Göllnitz-Gordon theorem.

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1. Introduction

Gordon's theorem [1] in the theory of partitions has an analytic counterpart due to Andrews [2][3], which reads as follows: For $k = 2, 3, 4, \ldots,\;\; s = 1, 2, \ldots, k$ (and $|q| < 1$, which is assumed to hold throughout the paper)

$$\prod_{\substack{n=1 \\ n \not\equiv 0, \pm s (\mathrm{mod}\;\; 2k+1)}}^{\infty} (1-q^n)^{-1} \;=\; \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_s+N_{s+1}+\ldots+N_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \quad , \qquad (1.1)$$

where

$$N_j \;=\; m_j + m_{j+1} + \ldots + m_{k-1} \qquad\qquad (N_k = 0), \qquad (1.2)$$

and

$$(z)_0 = 1 \;\; , \qquad (z)_m = (z;q)_m = \prod_{\ell=0}^{m-1}(1-zq^{\ell}) \qquad \text{for} \;\; m = 1, 2, 3, \ldots . \qquad (1.3)$$

To appreciate the broad context in which the Gordon-Andrews identities (1.1) play an important role, the reader is invited to look at [4] and references therein.

Using Jacobi's triple product identity (see *e.g.* eq. (2.2.10) in [3]), the lhs of (1.1) can be shown to be equal to $\chi_{1,s}^{(2,2k+1)}(q)$, where, more generally,

$$\chi_{r,s}^{(p,p')}(q) \;=\; \chi_{p-r,p'-s}^{(p,p')}(q) \;=\; \frac{1}{(q)_\infty} \sum_{\ell \in \mathbf{Z}} \left(q^{\ell(\ell pp' + rp' - sp)} - q^{(\ell p + r)(\ell p' + s)} \right) \qquad (1.4)$$

is [5] the (normalized) character of the irreducible highest weight representation of the Virasoro algebra at central charge $c^{(p,p')} = 1 - \frac{6(p'-p)^2}{pp'}$ and highest weight $\Delta_{r,s}^{(p,p')} = \frac{(rp'-sp)^2-(p'-p)^2}{4pp'}$. For given coprime integers p and p' $(p' > p \geq 2)$ these representations, with $r = 1, 2, \ldots, p-1$ and $s = 1, 2, \ldots, p'-1$, span the spectrum [6] of the minimal model $\mathcal{M}(p,p')$ of conformal field theory (CFT) [7].

Thus the left- and right-hand sides of eq. (1.1) constitute alternative expressions, referred to as product and fermionic forms, respectively, for the full set of characters of the minimal CFT $\mathcal{M}(2, 2k+1)$, for all $k \geq 2$. The form given in (1.4) is referred to as a bosonic (or free-field) form. The motivation for the "physical terminology" we use is reviewed in [8][9]. For the most recent developments in the subject of q-series identities, as inspired by work in two-dimensional physics, see [10] and references therein.

The aim of the present paper is to discuss supersymmetric analogs of the Gordon-Andrews identities, namely product-sum identities similar to (1.1) which provide alternative expressions for the characters of the minimal N=1 super-CFTs [11] $\mathcal{SM}(2, 4k)$. For

various reasons (see *e.g.* [12]) the family of theories $\mathcal{SM}(2,4k)$ is the natural supersymmetric analog of the family $\mathcal{M}(2,2k+1)$; this is also revealed by the fermionic forms we will encounter below for the characters $\hat{\chi}^{(2,4k)}_{1,s}(q)$ of the super-CFTs.

The (normalized) characters of a generic $N{=}1$ superconformal minimal model $\mathcal{SM}(p,p')$ are given by [13]

$$\hat{\chi}^{(p,p')}_{r,s}(q) = \hat{\chi}^{(p,p')}_{p-r,p'-s}(q) = \frac{(-q^{\varepsilon_{r-s}})_\infty}{(q)_\infty} \sum_{\ell\in\mathbf{Z}} \left(q^{\ell(\ell pp'+rp'-sp)/2} - q^{(\ell p+r)(\ell p'+s)/2} \right) , \quad (1.5)$$

where[1]

$$\varepsilon_a \;=\; \begin{cases} \frac{1}{2} & \text{if } a \text{ is even} \quad (\leftrightarrow \text{NS sector}) \\ 1 & \text{if } a \text{ is odd} \quad (\leftrightarrow \text{ R sector}) \end{cases} \quad (1.6)$$

Here $r=1,2,\ldots,p-1$ and $s=1,2,\ldots,p'-1$ as in the non-supersymmetric case, but this time $p'>p\geq 2$ are not necessarily coprime (in fact $\frac{p'-p}{2}$ and p must be coprime integers). Depending on the parity of $(r-s)$, the character (1.5) counts the multiplicities of weights in the irreducible highest-weight representation of the Neveu-Schwarz (NS) or Ramond (R) supersymmetric extension of the Virasoro algebra, with central charge $\hat{c}^{(p,p')}=\frac{3}{2}(1-\frac{2(p'-p)^2}{pp'})$ and highest weight $\hat{\Delta}^{(p,p')}_{r,s}=\frac{(rp'-sp)^2-(p'-p)^2}{8pp'}+\frac{2\varepsilon_{r-s}-1}{16}$.

Invoking Jacobi's triple product identity once again, the characters of $\mathcal{SM}(2,4k)$ are brought into the following product forms:

$$\begin{aligned}
\hat{\chi}^{(2,4k)}_{1,s}(q) \;&=\; \prod_{\substack{n=1\\ n\not\equiv 2 (\mathrm{mod}\ 4)\\ n\not\equiv 0,\pm s (\mathrm{mod}\ 4k)}}^{\infty} (1-q^{n/2})^{-1} && \text{for } s=1,3,\ldots,2k-1,\\
&=\; \prod_{\substack{n=1\\ n\ \mathrm{odd}}}^{\infty} (1-q^n)^{-1} \prod_{\substack{n=1\\ n\not\equiv 0,\pm s/2 (\mathrm{mod}\ 2k)}}^{\infty} (1-q^n)^{-1} && \text{for } s=2,4,\ldots,2k-2,\\
&=\; \prod_{\substack{n=1\\ n\not\equiv 0 (\mathrm{mod}\ k)}}^{\infty} \frac{1+q^n}{1-q^n} && \text{for } s=2k.
\end{aligned} \quad (1.7)$$

In this connection recall Euler's identity $(-q)_\infty = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$, which was also used in the derivation of (1.7).

Our main interest is in companion fermionic forms to the products in eq. (1.7). There exist at least two types of such forms, which will be presented in the first two subsections of section 2. The q-series identities conjectured there are supplemented in subsection

[1] In order to avoid subtleties related to the branches of the square root of q, we restrict ourselves from here on to $q\in[0,1)$ when $r-s$ is even.

2.3 by an even stronger conjecture involving two-variable power series. The two types of fermionic forms are associated with two families of thermodynamic Bethe Ansatz (TBA) systems. This observation, as discussed in section 3, helps us identify two different integrable perturbations of the models $\mathcal{SM}(2,4k)$ – one preserving and the other breaking the $N{=}1$ supersymmetry – as corresponding to these TBA systems. Section 4 contains some further comments and open questions.

2. q-series identities

2.1. First fermionic form.

The equality of the fermionic forms of the first type and the products on the first line of eq. (1.7) is an analytic counterpart [14] of a generalization of the Göllnitz-Gordon theorem (see section 7.4 of [3]). The fermionic sums read, for $s=1,3,\ldots,2k-1$,

$$\begin{aligned}\hat{\chi}^{(2,4k)}_{1,s}(q) &= \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{(-q^{1/2})_{N_1}\; q^{\frac{1}{2}N_1^2+N_2^2+\ldots+N_{k-1}^2+N_{(s+1)/2}+N_{(s+3)/2}+\ldots+N_{k-1}}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \\ &= \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_{(s+1)/2}+N_{(s+3)/2}+\ldots+N_{k-1}-N_1m_k+\frac{1}{2}m_k^2}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \begin{bmatrix} N_1 \\ m_k \end{bmatrix}_q \;.\end{aligned} \tag{2.1}$$

(In [3][14] only the case $s=2k-1$ is presented, with a misprint.) Here the N_j are as in (1.2), and the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}} & \text{if } \; 0\le m\le n \\ 0 & \text{otherwise ,} \end{cases} \tag{2.2}$$

for $m,n\in\mathbf{Z}$. The equality between the two lines of (2.1) is established with the aid of identity (3.3.6) of [3].

Eq. (2.1) provides fermionic forms for all the characters in the NS sector of $\mathcal{SM}(2,4k)$. In the R sector we are only able to conjecture similar forms for two characters, the ones

labeled by $s=2$ and $s=2k$:

$$\begin{aligned}
\hat{\chi}^{(2,4k)}_{1,2}(q) &= \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{(-q)_{N_1}\ q^{\frac{1}{2}N_1(N_1+1)+N_2(N_2+1)+\ldots+N_{k-1}(N_{k-1}+1)}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \\
&= \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{q^{N_1(N_1+1)+N_2(N_2+1)+\ldots+N_{k-1}(N_{k-1}+1)-N_1 m_k+\frac{1}{2}m_k(m_k-1)}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \begin{bmatrix} N_1 \\ m_k \end{bmatrix}_q , \\
\hat{\chi}^{(2,4k)}_{1,2k}(q) &= \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{(-1)_{N_1}\ q^{\frac{1}{2}N_1(N_1+1)+N_2^2+\ldots+N_{k-1}^2}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \\
&= \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2-N_1 m_k+\frac{1}{2}m_k(m_k+1)}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \begin{bmatrix} N_1 \\ m_k \end{bmatrix}_q .
\end{aligned} \tag{2.3}$$

These identities have been verified for various small values of $k>2$ up to high orders in the q-series, using Mathematica.

The case $k=2$ is already known. Eqs. (1.7),(2.3) reduce then to the identities

$$\begin{aligned}
\hat{\chi}^{(2,8)}_{1,2}(q) &= \prod_{\substack{n=1 \\ n\not\equiv 0 (\mathrm{mod}\ 4)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{(-q)_m\ q^{m(m+1)/2}}{(q)_m} \\
\hat{\chi}^{(2,8)}_{1,4}(q) &= \prod_{\substack{n=1 \\ n\ \mathrm{odd}}}^{\infty} \frac{1+q^n}{1-q^n} = \sum_{m=0}^{\infty} \frac{(-1)_m\ q^{m(m+1)/2}}{(q)_m} ,
\end{aligned} \tag{2.4}$$

which are equivalent to identities (8) and (12), respectively, on Slater's list [15]. For completeness we also write down explicitly the product-sum identities corresponding to the two NS characters of the model $\mathcal{SM}(2,8)$, derived from eqs. (1.7),(2.1) with $k=2$:

$$\begin{aligned}
\hat{\chi}^{(2,8)}_{1,1}(q) &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm1,\pm2 (\mathrm{mod}\ 8)}}^{\infty} (1-q^{n/2})^{-1} = \sum_{m=0}^{\infty} \frac{(-q^{1/2})_m\ q^{m(m+2)/2}}{(q)_m} \\
\hat{\chi}^{(2,8)}_{1,3}(q) &= \prod_{\substack{n=1 \\ n\not\equiv 0,\pm2,\pm3 (\mathrm{mod}\ 8)}}^{\infty} (1-q^{n/2})^{-1} = \sum_{m=0}^{\infty} \frac{(-q^{1/2})_m\ q^{m^2/2}}{(q)_m} .
\end{aligned} \tag{2.5}$$

These identities are equivalent to eqs. (34) and (36) in [15]. (Eqs. (38) and (39) of [15] provide further expressions, identified as $(-q^{1/2})_\infty \chi^{(3,4)}_{1,3}(-q^{1/2})$ and $(-q^{1/2})_\infty \chi^{(3,4)}_{1,1}(-q^{1/2})$ for the same products in (2.5); cf. also [16].) Eqs. (2.4)–(2.5) (with q replaced by q^2, perhaps) may be called supersymmetric analogs of the Rogers-Ramanujan-Schur identities; the latter two celebrated identities are obtained from (1.1) when specialized to the case k=2.

2.2. Second fermionic form.

The second type of fermionic forms for the characters of the same family of models $\mathcal{SM}(2,4k)$ is presented in the following conjecture:
For $k=2,3,4,\ldots$ and $s=1,2,\ldots,2k$

$$\begin{aligned}\hat{\chi}^{(2,4k)}_{1,s}(q) &= \sum_{m_1,\ldots,m_{2k-2}=0}^{\infty} \frac{q^{\frac{1}{2}(M_1^2+M_2^2+\ldots+M_{2k-2}^2)+M_s+M_{s+2}+\ldots+M_{2k-3}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{2k-2}}} \qquad (s \text{ odd}) \\ &= \sum_{m_1,\ldots,m_{2k-2}=0}^{\infty} \frac{q^{\frac{1}{2}(M_1^2+M_2^2+\ldots+M_{2k-2}^2)+M_s+M_{s+2}+\ldots+M_{2k-2}+\frac{1}{2}\tilde{M}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{2k-2}}} \qquad (s \text{ even}),\end{aligned} \tag{2.6}$$

where

$$M_j = m_j + m_{j+1} + \ldots + m_{2k-2} \ , \qquad \tilde{M} = m_1 + m_3 + \ldots + m_{2k-3} \ . \tag{2.7}$$

Again, this conjecture has been verified for various small $k>2$ and many orders in the expansion in powers of $q^{1/2}$. Evidence regarding the asymptotic behavior of high powers, namely the behavior of the q-series as $q \to 1^-$, will be presented in subsection 3.2.

For k=2 the fermionic form (2.6) can be brought by a simple change of the two summation variables into the fermionic form on the second lines of eqs. (2.1) and (2.3). (As noted above, the latter double-sums can be shown to be equal to the single-sums presented explicitly in eqs. (2.4)–(2.5).) Thus the identities (2.6) are proven in this case.

It is also worth noting that the special case s=3 (still k=2) has already been encountered before [17] in disguise. To demonstrate that, we first of all note that the N=1 super-CFT $\mathcal{SM}(2,8)$ is in fact equivalent to the model $\mathcal{M}(3,8)$ in the N=0 series. At the level of characters this equivalence is exhibited by the decomposition

$$\begin{aligned}&\hat{\chi}^{(2,8)}_{1,1}(q) = \chi^{(3,8)}_{1,1}(q) + q^{3/2}\chi^{(3,8)}_{1,7}(q) \ , \qquad \hat{\chi}^{(2,8)}_{1,3}(q) = \chi^{(3,8)}_{1,3}(q) + q^{1/2}\chi^{(3,8)}_{1,5}(q) \ , \\ &\hat{\chi}^{(2,8)}_{1,2}(q) = \chi^{(3,8)}_{1,4}(q) \ , \qquad \hat{\chi}^{(2,8)}_{1,4}(q) = \chi^{(3,8)}_{1,2}(q) + q\chi^{(3,8)}_{1,6}(q) = 2\chi^{(3,8)}_{1,2}(q) - 1 \ .\end{aligned} \tag{2.8}$$

Now in [17] fermionic expressions for the characters $\chi^{(3,3k+2)}_{1,k+1}$ (misprinted there as $\chi^{(3,3k+2)}_{1,k}$) were conjectured. In the case k=2 this conjecture reduces to the sum representation of $\hat{\chi}^{(3,8)}_{1,3}$ given in (2.6), with the summation restricted to run over $m_1 \in 2\mathbf{Z}$. Fermionic forms for the remaining $\chi^{(3,8)}_{r,s}$ as the sums (2.6) at k=2 with even/odd restrictions on m_1 can now be deduced as well using (2.8).

2.3. A stronger conjecture.

In section 7.2 of [3] the generating functions $J_{k,i}(a;x;q)$ are introduced. For $a = -q^{-1/2}$, in particular, the definition there leads to

$$
\begin{aligned}
J_{k,i}(-q^{-1/2};x;q) \; &= \; \sum_{n=0}^{\infty} (-x^k)^n q^{kn(n+1)-(i-\frac{1}{2})n} \\
&\times \; \frac{(-q^{1/2})_n \; \left(1+xq^{n+\frac{1}{2}}-(1+q^{n+\frac{1}{2}})x^i q^{(i-\frac{1}{2})(2n+1)}\right)(-xq^{n+\frac{3}{2}})_\infty}{(q)_n \; (xq^{n+1})_\infty} \; .
\end{aligned}
\tag{2.9}
$$

We now conjecture that the functions (2.9), with $i = 1,2,\ldots,k$, admit the following fermionic representations:

$$
\begin{aligned}
&J_{k,i}(-q^{-1/2};x;q) \\
&= \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{x^{N_1+N_2+\ldots+N_{k-1}}(-q^{1/2})_{N_1} \; q^{\frac{1}{2}N_1^2+N_2^2+\ldots+N_{k-1}^2+N_i+N_{i+1}+\ldots+N_{k-1}}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \\
&= \sum_{m_1,\ldots,m_{2k-2}=0}^{\infty} \frac{x^{M_1+M_3+\ldots+M_{2k-3}} \; q^{\frac{1}{2}(M_1^2+M_2^2+\ldots+M_{2k-2}^2)+M_{2i-1}+M_{2i+1}+\ldots+M_{2k-3}}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{2k-2}}} \; ,
\end{aligned}
\tag{2.10}
$$

where the N_j and M_j are defined in (1.2) and (2.7).

At $x = 1$ this conjecture reduces to the first equalities in (2.1) and (2.6), since it can easily be checked that for $s = 1,3,\ldots,2k-1$ eq. (2.9) yields $J_{k,(s+1)/2}(-q^{-1/2};1;q) = \hat{\chi}^{(2,4k)}_{1,s}(q)$, where the character is given by (1.5). Finally, let us note that the same generating functions $J_{k,i}(a;x;q)$ play a similar role also in the non-supersymmetric case. Namely (see eq. (7.3.8) in [3]), for $i = 1,2,\ldots,k$

$$
\begin{aligned}
J_{k,i}(0;x;q) \; &= \; \sum_{n=0}^{\infty} (-x^k)^n \; q^{(k+\frac{1}{2})n(n+1)-in} \; \frac{1-x^i q^{i(2n+1)}}{(q)_n \; (xq^{n+1})_\infty} \\
&= \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{x^{N_1+N_2+\ldots+N_{k-1}} \; q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_i+N_{i+1}+\ldots+N_{k-1}}}{(q)_{m_1}(q)_{m_2}\ldots(q)_{m_{k-1}}} \; ,
\end{aligned}
\tag{2.11}
$$

so that, in particular, $J_{k,i}(0;1;q) \; = \; \chi^{(2,2k+1)}_{1,i}(q)$.

3. Perturbed CFT, TBA systems, and dilogarithm sum-rules

The way we arrived at the results described in section 2 was by considering certain thermodynamic Bethe Ansatz (TBA) systems, which specify the finite-volume ground state

energy of some integrable massive quantum field theories in 1+1 dimensions as a solution of a set of nonlinear integral equations [18]. The volume dependence of the ground state energy provides information about the renormalization group flow of the massive theory from the CFT in the ultraviolet limit to the massive infrared fixed point. In particular, the ultraviolet effective central charge $\tilde{c} = c - 24\Delta_{\min}$, where c is the Virasoro central charge and $\Delta_{\min}$ is the lowest conformal dimension of the CFT, can be extracted from the TBA system and turns out to be given by a sum of Rogers dilogarithms. The conformal dimension Δ_p of the perturbing field, which "drives" the theory away from the ultraviolet fixed point along the renormalization group trajectory, can also be deduced from that system.

To make the above brief discussion slightly more concrete, let us first summarize some properties of a generic TBA system of the type of interest here (see *e.g.* [19]). The scaled finite-volume ground state energy is given, as a function of the scaled dimensionless volume ρ, by

$$e_0(\rho) \; = \; -\frac{\rho}{2\pi} \sum_{a=1}^{n} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \; \mu_a \cosh\theta \; \ln\left(1 + e^{-\epsilon_a(\theta)}\right) \quad , \tag{3.1}$$

where the $\epsilon_a(\theta)$ satisfy the equations

$$\epsilon_a(\theta) \; = \; \rho\mu_a \cosh\theta - \sum_{b=1}^{n} \left(K_{ab} * \ln(1 + e^{-\epsilon_b})\right)(\theta) \quad . \tag{3.2}$$

Here the μ_a are nonnegative mass parameters, the kernel $K_{ab}(\theta)$ is a symmetric matrix of even functions which depend on the S-matrix of the theory, and the convolution is defined by $(f * g)(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} f(\theta - \theta')g(\theta')$. The effective central charge obtained from the above TBA system is

$$\tilde{c} \; = \; -12e_0(0) \; = \; \frac{6}{\pi^2} \sum_{a=1}^{n} [\mathcal{L}(1 - x_a) - \mathcal{L}(1 - y_a)] \quad , \tag{3.3}$$

where (for $0 \leq z \leq 1$)

$$\mathcal{L}(z) \; = \; -\frac{1}{2} \int_0^z dt \Big[\frac{\ln t}{1 - t} + \frac{\ln(1 - t)}{t}\Big] \tag{3.4}$$

is the Rogers dilogarithm and the x_a, y_a are solutions to the equations

$$1 - x_a \; = \; \prod_{b=1}^{n} x_b^{B_{ab}} \quad , \qquad 1 - y_a \; = \; \sigma_a \prod_{b=1}^{n} y_b^{B_{ab}} \quad . \tag{3.5}$$

In (3.5), $\sigma_a = 1$ if $\mu_a = 0$ and 0 otherwise, and

$$B_{ab} \;=\; \delta_{ab} - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\; K_{ab}(\theta) \quad , \tag{3.6}$$

which in all known cases turn out to be rational numbers.

The relation between all that and fermionic forms of CFT characters is discussed in detail in [17]. Here we just repeat the upshot of that discussion. Namely, making use of part of the data in the previous paragraph, construct the fermionic sums

$$F_B(q) \;=\; \sum_{m_1,\ldots,m_n=0}^{\infty} q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t - \mathbf{A}\cdot\mathbf{m}} \prod_{a=1}^{n} \begin{bmatrix} (\mathbf{m}(1-B))_a + u_a \\ m_a \end{bmatrix}_q \quad , \tag{3.7}$$

where B is the matrix (3.6), $\mathbf{m} = (m_1, \ldots, m_n)$, $\mathbf{A}$ is a real n-dimensional vector, and the u_a are some more parameters; in particular $u_a = \infty$ if μ_a in (3.1)–(3.2) is strictly positive, and otherwise the u_a are such that the upper entries of all the q-binomials in (3.7) are integral for infinitely many $\mathbf{m}$ (note that $\begin{bmatrix} \infty \\ m \end{bmatrix}_q = \frac{1}{(q)_m}$, and we also use the convention that $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$ if $n \notin \mathbf{Z}$). Then, letting $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$ (with $\mathrm{Im}\tau > 0$), one has

$$F_B(q) \;\sim\; \tilde{q}^{-\tilde{c}/24} \qquad \text{as} \qquad q \to 1^- \quad , \tag{3.8}$$

with $\tilde{c}$ given by (3.3)–(3.6). This is the asymptotic behavior of the characters of a CFT whose effective central charge is $\tilde{c}$, as dictated by their modular properties.

The papers [17][20][21] suggest that associated with every integrable (relevant) perturbation of any rational CFT there is a set of fermionic sums of the type (3.7), containing at least one sum for each character of that CFT. This set is characterized by a fixed quadratic form B and by the list of the components of $\mathbf{u}$ which are infinite. The remaining components $u_a < \infty$ and the vector $\mathbf{A}$ depend on the specific character in question. In particular, in all cases studied it was observed that the vacuum character (corresponding to the representation of lowest highest-weight $\Delta_{\min}$ in the CFT) is represented by (3.7) with $\mathbf{A} = 0$ and all $u_a = 0$, whenever $u_a < \infty$ (and perhaps some restrictions on the vectors $\mathbf{m}$ which are summed over).

3.1. First family of TBA systems.

The TBA systems which are relevant for the discussion in section 2.1 are obtained by "folding in half" the systems derived in [22] for the most relevant SUSY-preserving perturbations of the even members in the series of minimal N=2 super-CFTs, described by the Landau-Ginzburg superpotential $\frac{X^{2k}}{2k} - \lambda X$ $(k = 2, 3, 4, \ldots)$. This folding is the supersymmetric analog of the procedure through which the TBA systems of[2] $\mathcal{M}(2, 2k+1) + \phi_{1,3}$ are related to the systems of the theories of $\mathbf{Z}_{2k-1}$ parafermions perturbed by the most relevant thermal operator [18].

Specifically, in both cases the $\mathbf{Z}_2 \times \mathbf{Z}_{2k-1}$ symmetry of the unfolded model is reflected in the symmetries $\mu_a = \mu_{n+1-a}$ and $K_{ab}(\theta) = K_{n+1-a,n+1-b}(\theta)$ of the data in (3.2), where $(n = 2k-2)$ $n = 2k$ in the (non-)supersymmetric case. As a result, the solution of (3.2) satisfies $\epsilon_a(\theta) = \epsilon_{n+1-a}(\theta)$, and so one finds that $e_0(\rho) = 2e_0^{(2)}(\rho)$, where $e_0^{(2)}(\rho)$ is the scaled ground state energy of the folded TBA system

$$
\begin{aligned}
e_0^{(2)}(\rho) \;&=\; -\frac{\rho}{2\pi} \sum_{a=1}^{n/2} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, \mu_a \cosh\theta\ \ln\left(1 + e^{-\epsilon_a(\theta)}\right) \\
\epsilon_a(\theta) \;&=\; \rho\mu_a \cosh\theta - \sum_{b=1}^{n/2} \left(K_{ab}^{(2)} * \ln(1 + e^{-\epsilon_b})\right)(\theta) \quad .
\end{aligned}
\tag{3.9}
$$

Here

$$
K_{ab}^{(2)}(\theta) \;=\; K_{ab}(\theta) + K_{a,n+1-b}(\theta) \qquad\qquad (a, b = 1, 2, \ldots, \frac{n}{2}). \tag{3.10}
$$

The analogy with the non-supersymmetric case suggests that the $n = k$ TBA system, obtained by folding the N=2 supersymmetric system of [22] with $n = 2k$, gives the finite-volume ground state energy of the N=1 super-CFT $\mathcal{SM}(2, 4k)$ perturbed by the top component $\hat{\phi}_{1,3}^{\text{top}}$ of the superfield $\hat{\phi}_{1,3}$ (so that the corresponding conformal dimension is $\Delta_p^{(2)} = \hat{\Delta}_{1,3}^{(2,4k)} + \frac{1}{2} = \frac{1}{2k}$). This folded TBA system can in principle be – but has not yet been – derived from the (nondiagonal) factorizable scattering theory proposed in [12] to describe this N=1 SUSY-preserving perturbation.

If correct, our proposal implies that the scaled ground state energy of $\mathcal{SM}(2, 4k) + \hat{\phi}_{1,3}^{\text{top}}$ is precisely half that of the corresponding perturbed N=2 model, for all $\rho \geq 0$. Two simple consistency tests support the validity of this relation. First, the effective central

[2] Henceforth we use the symbol '$\mathcal{M} + \phi_p$' as a shorthand for the phrase 'the ϕ_p-perturbation of the CFT model $\mathcal{M}$'.

charge $\tilde{c} = \hat{c}^{(2,4k)} - 24\hat{\Delta}^{(2,4k)}_{1,2k-1} = \frac{3(k-1)}{2k}$ of $\mathcal{SM}(2,4k)$ is indeed half the one $\tilde{c} = c = \frac{3\cdot(2k-2)}{(2k-2)+2} = \frac{3(k-1)}{k}$ of the N=2 super-CFT. And second, note that in the N=2 super-CFT the three-point function $\langle\phi_{\rm min}\phi_p\phi_{\rm min}\rangle$ vanishes, being equal to $\langle\phi_p\rangle = 0$, while in the N=1 theory it does not, as $\langle\phi_{\rm min}\phi_p\phi_{\rm min}\rangle \propto \langle\hat{\phi}^{(2,4k)}_{1,2k-1}\hat{\phi}^{(2,4k)}_{1,3}\hat{\phi}^{(2,4k)}_{1,2k-1}\rangle \neq 0$. Consequently [18], the relation $2(1-\Delta_p) = 1-\Delta_p^{(2)}$ between the dimensions of the corresponding perturbing fields must hold; and indeed it does, since $\Delta_p = \frac{1}{2} + \frac{1}{2\cdot 2k} = \frac{2k+1}{4k}$ in the N=2 theory, while $\Delta_p^{(2)} = \frac{1}{2k}$.

Finally, we point out that the fermionic forms on the second lines of eqs. (2.1) and (2.3) follow the construction (3.7) described above. To see that, note that these fermionic sums are of the form (3.7), with $n = k$, $u_a = \infty$ for $a = 1, 2\ldots, k-1$, and B given by

$$\begin{aligned} &B_{ab} = (2C^{-1}_{T_{k-1}})_{ab} = 2\max(a,b) && (a,b = 1,2,\ldots,k-1), \\ &B_{ak} = B_{ka} = 2\delta_{ak} - 1 && (a = 1,2,\ldots,k), \end{aligned} \tag{3.11}$$

where C_{T_n} is the Cartan matrix of the tadpole diagram $A_{2n}/\mathbf{Z}_2$ (see *e.g.* [17]). This is precisely what one obtains when plugging (3.10) into (3.6), where the $K_{ab}(\theta)$ are specified in [22] (the labels $a = 0, \bar{0}$ in the latter reference correspond to $a = k, k+1$ in our notation). The resulting dilogarithm sum-rule for the effective central charge of the CFT $\mathcal{SM}(2,4k)$, which follows from [22], reads

$$\frac{6}{\pi^2}\left[\sum_{a=1}^{k-1}\mathcal{L}\left(\frac{\sin^2\frac{\pi}{2k}}{\sin^2\frac{(2a+1)\pi}{4k}}\right) + \mathcal{L}\left(1 - \frac{1}{4\cos^2\frac{\pi}{4k}}\right) - \mathcal{L}\Big(\frac{1}{2}\Big)\right] = \frac{3(k-1)}{2k} \ . \tag{3.12}$$

3.2. Second family of TBA systems.

We now briefly discuss a second family of TBA systems which is related to the second type of fermionic forms for the characters of the super-CFTs $\mathcal{SM}(2,4k)$, presented in subsection 2.2. The systems in question are denoted by $T_{2k-2} \diamond T_1$ in the notation of [23][24]; the reader is referred to these references for further details. The effective central charge extracted from this system is $\tilde{c} = \frac{3(k-1)}{2k}$ (using eq. (40) of [24] with $r = 2k-2$ and $h = 4k-3$, as appropriate for T_{2k-2}). Furthermore, using the so-called periodicity $P = \frac{4k}{4k-3}$ of the system we read off the conformal dimension of the perturbing field $\Delta_p = 1 - \frac{2}{P} = -\frac{2k-3}{2k}$, which applies when the three-point function $\langle\phi_{\rm min}\phi_p\phi_{\rm min}\rangle$ in the ultraviolet CFT is nonvanishing.

Noting that the effective central charge is that of $\mathcal{SM}(2,4k)$, and that $\Delta_p = \hat{\Delta}^{(2,4k)}_{1,5}$ is the dimension of the bottom component $\hat{\phi}^{\rm bot}_{1,5}$ of the superfield $\hat{\phi}_{1,5}$ in this CFT (and that

indeed $\langle \phi_{\min}\phi_p\phi_{\min}\rangle \propto \langle \hat{\phi}^{(2,4k)}_{1,2k-1}\hat{\phi}^{(2,4k)}_{1,5}\hat{\phi}^{(2,4k)}_{1,2k-1}\rangle \neq 0$), we propose that the TBA system $T_{2k-2}\diamond T_1$ is associated with $\mathcal{SM}(2,4k)+\hat{\phi}^{\rm bot}_{1,5}$. An indication for the integrability of this SUSY-breaking perturbation is revealed by applying Zamolodchikov's counting argument [25], which shows the existence of a nontrivial integral of motion of spin s=3 for all k (for k=2 also s=5,7 are found).[3]

The proposal above is apparently new for all $k>2$. In the case k=2 it is consistent with the identification noted in [24] of the system $T_2\diamond T_1$ as corresponding to $\mathcal{M}(3,8)+\phi_{1,3}$; consistency is ensured here by the equivalence (see subsection 2.2) of the latter theory with $\mathcal{SM}(2,8)+\hat{\phi}^{\rm bot}_{1,5}=\mathcal{SM}(2,8)+\hat{\phi}^{\rm bot}_{1,3}$.

For completeness, let us further suggest that the TBA system $T_{2k-1}\diamond T_1$ with $k\geq 2$ is associated with the perturbation of the tensor-product CFT $\mathcal{M}(3,4)\otimes\mathcal{M}(2,2k+1)$ by the operator $\phi^{(3,4)}_{1,3}\otimes\phi^{(2,2k+1)}_{1,2}$.[4] This identification is supported by the match between the effective central charge and the dimension Δ_p of the perturbation in this theory with the ones deduced from the TBA system. (Note that in this case $\langle\phi_{\min}\phi_p\phi_{\min}\rangle = \langle\phi^{(3,4)}_{1,3}\rangle\langle\phi^{(2,2k+1)}_{1,k}\phi^{(2,2k+1)}_{1,2}\phi^{(2,2k+1)}_{1,k}\rangle=0$, and therefore Δ_p is related to the periodicity P of the system according to the choice $\Delta_p=1-\frac{1}{P}$ in eq. (3.18) of [23].)

The TBA systems $T_n\diamond T_1$ are completely massive, in the sense that all $\mu_a>0$ in (3.2), and hence the corresponding factorizable scattering theories have diagonal S-matrices. Their scattering amplitudes are given in [24]. Thus the proposals above also answer the question which perturbed CFTs all these diagonal S-matrix theories describe.

Finally, it is straightforward to see that the fermionic sums (2.6) are of the type obtained through the construction (3.7) based on the system $T_{2k-2}\diamond T_1$: it follows from (3.6) and [23][24] that the matrix B in this case is $B=C^{-1}_{T_{2k-2}}\otimes C_{T_1}=C^{-1}_{T_{2k-2}}$ (*i.e.* $B_{ab}=\max(a,b)$, and so $\frac{1}{2}\mathbf{m}B\mathbf{m}^t=\frac{1}{2}(M_1^2+M_2^2+\ldots+M_{2k-2}^2)$), and u_a=∞ for all $a=1,2,\ldots,2k-2$. The corresponding dilogarithm sum-rule for $\tilde{c}$ is a special case of the general results in [23][24]. The implied asymptotic behavior (3.8) of the conjectured fermionic forms in (2.6) provides further support for their correctness.

---

[3] The $\hat{\phi}^{\rm bot}_{1,5}$-perturbation is not included in the set of integrable perturbations of $\mathcal{SM}(p,p')$ discussed in [26], perhaps because it is irrelevant in the case of the unitary series $\mathcal{SM}(p,p+2)$, namely $\hat{\Delta}^{(p,p+2)}_{1,5}>1$.

[4] The first model in this series, with k=2, is equivalent to $\mathcal{M}(5,12)+\phi_{2,7}$ (with the E_6 modular-invariant partition function [27] for $\mathcal{M}(5,12)$).

4. Comments and open questions

4.1. Regarding proofs.

There are several ways to approach the problem of proving the conjectures in section 2 (see [3][4] and references therein for some of the known methods). An approach which has a physical interpretation is based on a remarkable connection between q-series representing CFT characters and one-dimensional configuration sums encountered in corner transfer matrix computations in exactly solvable lattice models [28]. In the latter framework the infinite q-series arise as limiting cases of families of polynomials, termed finitized characters in [9]. It is sometimes easier to prove the stronger result of equality between finitized versions of bosonic and fermionic sum forms of the characters, from which the q-series identities follows [9][29-33].

I believe one can find and prove polynomial identities which imply the q-series identities conjectured in section 2. It would be interesting to see whether the corresponding polynomials are in fact one-dimensional configuration sums of any solvable lattice models.

4.2. Combinatorial interpretation.

As mentioned in the introduction, q-series identities of the type discussed in this paper can be viewed as analytic counterparts of combinatorial theorems in the theory of partitions [3]. In particular, according to theorem 7.11 in [3] and the first line of eq. (1.7), the Neveu-Schwarz characters $\hat{\chi}^{(2,4k)}_{1,2i-1}$ $(i = 1, 2, \ldots, k)$ are the generating functions

$$\hat{\chi}^{(2,4k)}_{1,2i-1}(q) \;=\; \sum_{n=0}^{\infty} D_{k,i}(n)\; q^{n/2} \qquad\qquad (i = 1, 2, \ldots, k) \tag{4.1}$$

of the number of partitions $D_{k,i}(n)$ of $\frac{n}{2} \in \frac{1}{2}\mathbf{Z}_{\geq 0}$ in the form $\frac{n}{2} = b_1 + b_2 + \ldots + b_m$ $(b_j \in \frac{1}{2}\mathbf{Z}_{\geq 1})$ in which no half-odd-integral part is repeated, $b_j \geq b_{j+1}$, $b_j - b_{j+k-1} \geq 1$ if $b_j \in \mathbf{Z} + \frac{1}{2}$, $b_j - b_{j+k-1} > 1$ if $b_j \in \mathbf{Z}$, and at most $i-1$ parts are ≤ 1.

The first interesting problem which arises is to show that the fermionic forms in eq. (2.1) and on the first line of eq. (2.6) can be directly interpreted as the generating functions of the coefficients $D_{k,i}(n)$ (analogous result pertaining to the rhs of eq. (1.1) is found in [34]). It is furthermore interesting to explore possible combinatorial interpretations of the Ramond characters $\hat{\chi}^{(2,4k)}_{1,2i}$, for which fermionic forms are conjectured in (2.3) and the second line of (2.6).

Next, by analogy with the results of [35] (theorem 3.6) in the non-supersymmetric case, one may suspect that the combinatorial interpretation described above for the Neveu-Schwarz characters has the following representation-theoretical consequence. Let $v_{k,i}$ be a highest-weight state of conformal weight $\hat{\Delta}^{(2,4k)}_{1,2i-1}$ in the Verma module of the Neveu-Schwarz supersymmetrically extended Virasoro algebra at central charge $\hat{c}^{(2,4k)}$. Then the set of states

$$W_{-b_1} W_{-b_2} \dots W_{-b_m} v_{k,i} \tag{4.2}$$

form a basis for the *irreducible* highest-weight representation. Here $W_b = L_b$ if $b \in \mathbf{Z}$, $W_b = G_b$ if $b \in \mathbf{Z} + \frac{1}{2}$ (using standard notation for the generators of the N=1 super-Virasoro algebra), and the b_j are as in the above definition of the $D_{k,i}(n)$. To prove this statement, it is sufficient – by virtue of (4.1) – to show that the set (4.2) is linearly independent modulo singular states in the Verma module (in this connection the results of [36] are important).

4.3. Some more identities.

In subsection 2.1, eqs. (2.1) and (2.3), we noted the equality of fermionic k-multiple sums of the form (3.7) (which according to [17] carry a natural physical interpretation in terms of fermionic quasiparticles) with $(k-1)$-multiple sums of a slightly modified form. In particular, in the case of k=2 the modified single-sums, representing the characters (2.4)–(2.5) of the N=1 super-CFT $\mathcal{SM}(2,8)$, appear on Slater's list [15] in the product-sum identities (8), (12), (34), and (36). It is amusing to go through the list and try to find and interpret other modified fermionic sums as representing characters of some super-CFTs.

We found two sets of such sums, described below. Curiously, together with the model $\mathcal{SM}(2,8)$ the corresponding two super-CFTs $\mathcal{SM}(3,5)$ and $\mathcal{SM}(3,7)$ exhaust all the N=1 supersymmetric minimal models which are equivalent to non-supersymmetric ones:

$$\mathcal{SM}(2,8) \simeq \mathcal{M}(3,8) \quad , \quad \mathcal{SM}(3,5) \simeq \mathcal{M}(4,5) \quad , \quad \mathcal{SM}(3,7) \simeq \mathcal{M}_{E_6}(7,12) \tag{4.3}$$

(the subscript in the last model indicates the type of modular invariant partition function, according to the classification in [27]). The decomposition of characters of the two super-CFTs, analogous to (2.8) for $\mathcal{SM}(2,8)$, reads

$$\begin{aligned} \hat{\chi}^{(3,5)}_{1,1} &= \chi^{(4,5)}_{1,1} + q^{3/2}\chi^{(4,5)}_{1,4} \quad , \qquad & \hat{\chi}^{(3,5)}_{1,3} &= \chi^{(4,5)}_{1,2} + q^{1/2}\chi^{(4,5)}_{1,3} \quad , \\ \hat{\chi}^{(3,5)}_{1,2} &= \chi^{(4,5)}_{2,2} \quad , & \hat{\chi}^{(3,5)}_{1,4} &= \chi^{(4,5)}_{2,1} \quad , \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\hat{\chi}^{(3,7)}_{1,1} &= \chi^{(7,12)}_{1,1} + q^{3/2}\chi^{(7,12)}_{1,5} + q^{4}\chi^{(7,12)}_{1,1} + q^{25/2}\chi^{(7,12)}_{1,11} \quad , \\
\hat{\chi}^{(3,7)}_{1,3} &= \chi^{(7,12)}_{3,5} + q^{1/2}\chi^{(7,12)}_{3,7} + q^{5/2}\chi^{(7,12)}_{3,1} + q^{5}\chi^{(7,12)}_{3,11} \quad , \\
\hat{\chi}^{(3,7)}_{1,5} &= \chi^{(7,12)}_{2,5} + q^{1/2}\chi^{(7,12)}_{2,1} + q^{3/2}\chi^{(7,12)}_{2,7} + q^{8}\chi^{(7,12)}_{2,11} \quad , \\
\hat{\chi}^{(3,7)}_{1,2} &= \chi^{(7,12)}_{2,4} + q^{3}\chi^{(7,12)}_{2,8} \quad , \\
\hat{\chi}^{(3,7)}_{1,4} &= \chi^{(7,12)}_{3,4} + q\chi^{(7,12)}_{3,8} \quad , \\
\hat{\chi}^{(3,7)}_{1,6} &= \chi^{(7,12)}_{1,4} + q^{5}\chi^{(7,12)}_{1,8} \quad .
\end{aligned} \tag{4.5}$$

Now substituting in (4.4) the fermionic forms of the characters $\chi^{(4,5)}_{r,s}$, which were conjectured in [17] and proven in [9] (see also [31][37]), simple use of identity (3.3.6) of [3] leads to

$$\begin{aligned}
\hat{\chi}^{(3,5)}_{1,1}(q) &= \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q^{1/2})_{m/2}\; q^{3m^2/8}}{(q)_m} \quad , \\
\hat{\chi}^{(3,5)}_{1,3}(q) &= \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q^{1/2})_{m/2}\; q^{m(3m-4)/8}}{(q)_m} = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q^{1/2})_{(m+1)/2}\; q^{3(m^2-1)/8}}{(q)_m} \quad , \\
\hat{\chi}^{(3,5)}_{1,2}(q) &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q)_{(m-1)/2}\; q^{(m-1)(3m-1)/8}}{(q)_m} = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q)_{m/2}\; q^{m(3m-2)/8}}{(q)_m} \quad , \\
\hat{\chi}^{(3,5)}_{1,4}(q) &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q)_{(m-1)/2}\; q^{3(m^2-1)/8}}{(q)_m} \quad .
\end{aligned} \tag{4.6}$$

The first sums in each of the four lines of (4.6) appear in eqs. (100), (95), (62), and (63) of [15], respectively. To see that, replace q by q^2 in the first two lines of (4.6), change the summation variable m to $2n$ $(2n-1)$ when m is even (odd), and correct the misprint in eq. (100) of [15], where the factor $(-q^2; q^2, n)$ should read $(-q; q^2, n)$. The equalities between the pair of sums on lines two and three of (4.6) is apparently not noted in [15]. Product forms for all the characters of $\mathcal{SM}(3,5)$ can be inferred from the corresponding identities in [15] (see also [38]).

Turning next to the model $\mathcal{SM}(3,7)$, we speculate that the sums in eqs. (118), (117), and (119) of [15], respectively, provide the following fermionic forms for the three Neveu-

Schwarz characters of this model:

$$
\begin{aligned}
\hat{\chi}^{(3,7)}_{1,1}(q) &= \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q^{1/2})_{m/2}\ q^{m(m+4)/8}}{(q)_m} \quad , \\
\hat{\chi}^{(3,7)}_{1,3}(q) &= \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q^{1/2})_{m/2}\ q^{m^2/8}}{(q)_m} \quad , \\
\hat{\chi}^{(3,7)}_{1,5}(q) &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q^{1/2})_{(m+1)/2}\ q^{(m-1)(m+3)/8}}{(q)_m} \quad ,
\end{aligned}
\tag{4.7}
$$

while the sums in eqs. (81), (80), and (82) are equal to the three Ramond characters:

$$
\begin{aligned}
\hat{\chi}^{(3,7)}_{1,2}(q) &= \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{(-q)_{m/2}\ q^{m(m+2)/8}}{(q)_m} \ , \\
\hat{\chi}^{(3,7)}_{1,4}(q) &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q)_{(m-1)/2}\ q^{(m^2-1)/8}}{(q)_m} \ , \\
\hat{\chi}^{(3,7)}_{1,6}(q) &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-q)_{(m-1)/2}\ q^{(m-1)(m+5)/8}}{(q)_m} \quad .
\end{aligned}
\tag{4.8}
$$

One way to prove this would be to show the equality of the product forms implied by the identities in [15] listed above and the free-field forms (1.5) of the corresponding characters. Unfortunately, unlike in other cases, it seems that a more powerful tool than Jacobi's triple product identity is required for that purpose.

4.4. Some more TBA systems.

Like [17][20], section 3 demonstrates the fruitful interplay between q-series identities related to CFT characters and TBA systems. Here we would like to describe some further observations regarding TBA systems of the type encountered in subsection 3.2, with a few comments on their relation to q-series. Again, we use the notation introduced in [23].

The family of TBA systems $T_n \diamond T_1$ discussed in subsection 3.2 can be embedded in the general set of systems $(T_n \diamond T_k)_\ell$. Here $n, k = 1, 2, \ldots$ and $\ell = 1, 2, \ldots, k$; when k=1 the redundant index ℓ=1 is suppressed. The identification of the perturbed CFTs corresponding to these systems, as well as to their "unfolded" partners $(A_{2n} \diamond T_k)_\ell$ (see

subsection 3.1), was not attempted in [23][24][39].[5] Using the dilogarithm sum-rules in [39], the following formula for the effective central charges of the ultraviolet CFTs is obtained:

$$\begin{aligned}\tilde{c}((T_n \diamond T_k)_\ell) &= \tfrac{1}{2}\tilde{c}((A_{2n} \diamond T_k)_\ell) \\ &= n\left(\frac{k(2k+1)}{2(n+k+1)} - \frac{\ell(\ell-1)}{2n+\ell+1} - \frac{(k-\ell)(2k-2\ell+1)}{2(n+k-\ell+1)}\right) \quad .\end{aligned} \tag{4.9}$$

In addition, the conformal dimension of the perturbing field, as obtained from the periodicity $P = \frac{2(n+k+1)}{2n+1}$ of the system, is

$$\Delta_p = \begin{cases} 1 - \frac{1}{P} = \frac{2k+1}{2(n+k+1)} & \text{if} \quad \langle \phi_{\min}\phi_p\phi_{\min}\rangle = 0, \\ 1 - \frac{2}{P} = \frac{k-n}{n+k+1} & \text{if} \quad \langle \phi_{\min}\phi_p\phi_{\min}\rangle \neq 0. \end{cases} \tag{4.10}$$

A complete analysis of the general (n, k, ℓ) case seems formidable. In the sequel we extend the system identifications proposed in subsection 3.2 in a few special cases:

- $A_{2n} \diamond T_1$: Here we suspect that the unperturbed CFT is unitary, implying that $c = \tilde{c} = \frac{3n}{n+2}$ and that the first choice in eq. (4.10) applies, yielding $\Delta_p = \frac{3}{2(n+2)}$. The corresponding CFT is presumably a special point along the continuous line of theories obtained from the $SU(2)_n$ WZW model by a marginal perturbation. In particular, for n=1 we find c=1 and $\Delta_p = \frac{1}{2}$, and the corresponding finite-volume ground state energy is that of two free massive particles of equal masses; the unperturbed CFT is the so-called free Dirac point along the c=1 gaussian line. This is consistent with the fact that the folded system $T_1 \diamond T_1$ is associated [24] with the thermal perturbation of the Ising model.
- $(T_1 \diamond T_k)_1$: In this case $\tilde{c} = 1 - \frac{6}{2(k+1)(k+2)}$, identified as the effective central charge of the minimal model $\mathcal{M}(k+2, 2k+2)$ when k is odd, and $\mathcal{M}(k+1, 2k+4)$ when k is even. Using the first (second) choice in (4.10) for odd (even) k, the perturbing field is identified as $\phi_{2,1}^{(k+2,2k+2)}$ ($\phi_{1,5}^{(k+1,2k+4)}$). It can be checked that these choices are consistent with the condition on the three-point function stated in (4.10), using the fact that $\phi_{\min}^{(k+2,2k+2)} = \phi_{(k+1)/2,k}^{(k+2,2k+2)}$ ($\phi_{\min}^{(k+1,2k+4)} = \phi_{k/2,k+1}^{(k+1,2k+4)}$) for k odd (even) and the known fusion rules of the minimal models.

Of special interest is the case k=2, corresponding to $\mathcal{M}(3,8) + \phi_{1,5}$. Omitting the technical details, we state that the system $(T_1 \diamond T_2)_1$ can be shown to be equivalent to the k=2 folded TBA system in subsection 3.1. (By the same token, the unfolded system

[5] In [24] attention was restricted to $(G \diamond T_k)_\ell$ with h_G even, whereas in our case $h_{A_{2n}} = h_{T_n} = 2n+1$ is odd.

derived in [22] for the most relevant perturbation of the second model in the $N{=}2$ minimal series is just $(A_2 \diamond T_2)_1$.) A hint to this fact is seen by noting that for $k{=}2$ the matrix B of (3.11) is simply $C_{T_2} = (C_{T_1})^{-1} \otimes C_{T_2}$. Now the system in subsection 3.1 was associated with $\mathcal{SM}(2,8) + \hat{\phi}^{\rm top}_{1,3}$. As seen from (2.8), this theory is indeed equivalent to $\mathcal{M}(3,8) + \phi_{1,5}$, in line with our identification here.

As another special model in the series we note that $(T_1 \diamond T_5)_1$ corresponds to $\mathcal{M}(7,12) + \phi_{2,1}$. Using the equivalence in (4.3) this perturbed theory is related to $\mathcal{SM}(3,7) + \hat{\phi}^{\rm top}_{1,5}$. The TBA system for another SUSY-preserving perturbation of the same model $\mathcal{SM}(3,7)$, namely by $\hat{\phi}^{\rm top}_{1,3}$, is identified in [23] as $(A_1 \diamond T_2)_2$. When used in the construction (3.7), the latter system gives rise to the fermionic forms (4.7)–(4.8) for the characters of $\mathcal{SM}(3,7)$ (once the factors $(-q^{\epsilon})_n$ in (4.7)–(4.8) are expanded using eq. (3.3.6) of [3]).

- $(T_1 \diamond T_k)_2$ with odd $k \geq 3$: Here (4.9) gives $\tilde{c} = \frac{3}{5} + (1 - \frac{6}{k(k+2)})$, leading to the identification of the unperturbed CFT as $\mathcal{M}(3,5) \otimes \mathcal{M}(k,k+2)$. Furthermore, $\Delta_p = \frac{2k+1}{2(k+2)} = \Delta^{(3,5)}_{2,1} + \Delta^{(k,k+2)}_{1,2}$ using the first choice in (4.10), which is indeed the appropriate one for the perturbing field $\phi^{(3,5)}_{2,1} \otimes \phi^{(k,k+2)}_{1,2}$.

To provide stronger support for the above pairing of TBA systems and perturbed CFTs, as well as those in section 3, one must compare the (numerical) solution of the TBA systems with results for the finite-volume ground state energies obtained from conformal perturbation theory, as in [18]. Another direction for future work based on the discussion in the present subsection, is to look for fermionic forms for the characters of the encountered unperturbed CFTs, following the general construction (3.7).

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